Exam II Review - Worksheet Math 1131 Class #31

Name: Solutions Section: ____________

Taking [redacted] Derivatives

1. Compute the derivative of $f(x) = \sin(x^2 + x + 1)$

chain rule

$$f'(x) = \frac{d}{dx}\left(\sin(x^2+x+1)\right) = \cos(x^2+x+1)\cdot\frac{d}{dx}(x^2+x+1)$$

$$= \cos(x^2+x+1)\cdot(2x+1)$$

2. Compute the derivative of $f(x) = \cos(x^2) \bullet \sin(x^2)$

Product rule THEN chain rule

$(fg)' = fg' + gf'$

$$f'(x) = \cos(x^2)\cdot\frac{d}{dx}\left(\sin(x^2)\right) + \sin(x^2)\cdot\frac{d}{dx}\left(\cos(x^2)\right)$$

$$= \cos(x^2).\ \cos(x^2)\cdot 2x + \sin(x^2)\cdot(-\sin(x^2))\cdot 2x$$

$$= 2x\left(\cos^2(x^2) - \sin^2(x^2)\right)$$

3. Compute the derivative of $f(x) = \sin(x \cdot e^x)$

chain rule THEN product rule

$$f'(x) = \cos(x\cdot e^x)\cdot\frac{d}{dx}(x\cdot e^x)$$

$$= \cos(x\cdot e^x)\cdot\left(x\cdot\frac{d}{dx}e^x + e^x\cdot\frac{d}{dx}x\right)$$

$$= \cos(x\cdot e^x)\cdot(x\cdot e^x + e^x) = \cos(x\cdot e^x)\cdot e^x\cdot(x+1)$$

4. Compute the derivative of $f(x) = \sin(e^{x^2+x+1})$

double chain rule

$$f'(x) = \cos\left(e^{x^2+x+1}\right)\cdot\frac{d}{dx}\left(e^{x^2+x+1}\right)$$

$$= \cos\left(e^{x^2+x+1}\right)\cdot e^{x^2+x+1}\cdot\frac{d}{dx}(x^2+x+1)$$

$$= \cos\left(e^{x^2+x+1}\right)\cdot e^{x^2+x+1}\cdot(2x+1)$$

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Problems 5–7 have clever simple ways to solve them, but I illustrate "implicit differentiation" below.

5. Suppose that $x \cdot y = x^2$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x \cdot y) = \frac{d}{dx}(x^2)$$

$$x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = 2x$$

$$x \cdot y' + y = 2x$$

$$x \cdot y' = 2x - y$$

$$y' = \frac{2x-y}{x}$$

6. Suppose that $y^2 + 2xy + x^2 = 1$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^2 + 2xy + x^2\right) = \frac{d}{dx}(1)$$

$$2 \cdot y \cdot y' + \frac{d}{dx}(2 \cdot x \cdot y) + 2 \cdot x = 0$$

$$2 \cdot y \cdot y' + 2 \cdot x \cdot \frac{d}{dx}(y) + 2 \cdot y \cdot \frac{d}{dx}(x) + 2x = 0$$

$$2 \cdot y \cdot y' + 2 \cdot x \cdot y' + 2y + 2x = 0$$

$$2 \cdot y \cdot y' + 2 \cdot x \cdot x' = -2y - 2$$

$$y'(2y + 2x) = -2y - 2x$$

$$y' = \frac{-2y-2x}{2y+2x} = -1$$

7. Suppose that $e^y = \cos(x)$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx} e^y = \frac{d}{dx} \cos(x)$$

$$e^y \cdot y' = -\sin(x)$$

$$y' = \frac{-\sin(x)}{e^y}$$

8. Suppose that $e^y = \cos(y) + x$. Find $\frac{dy}{dx}$.

$$\frac{d}{dx} e^y = \frac{d}{dx}\left(\cos(y) + x\right)$$

$$e^y \cdot y' = -\sin(y) \cdot y' + 1$$

$$e^y \cdot y' + \sin(y) \cdot y' = 1$$

$$y'\left(e^y + \sin(y)\right) = 1$$

$$y' = \frac{1}{e^y + \sin(y)}$$

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When taking the derivative of a function containing logarithms, try using the 3 laws of logarithms to simplify first.

9. Let $f(x) = \ln\left(\sqrt{\dfrac{x^2+4x+4}{x^2-1}}\right)$. Find $f'(x)$

$$f(x) = \ln\left(\left(\frac{x^2+4x+4}{x^2-1}\right)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}\cdot \ln\left(\frac{x^2+4x+4}{x^2-1}\right)$$

$$= \frac{1}{2}\cdot\left(\ln(\underbrace{x^2+4x+4}_{(x+2)^2}) - \ln(x^2-1)\right)$$

$$f(x) = \frac{1}{2}\cdot 2\cdot\ln(x+2) - \frac{1}{2}\ln(x^2-1)$$

$$= \ln(x+2) - \frac{1}{2}\cdot\ln(x^2-1)$$

$$f'(x) = \frac{1}{x+2} - \frac{1}{2}\frac{1}{(x^2-1)}\cdot 2x$$

$$\boxed{f'(x) = \frac{1}{x+2} - \frac{x}{x^2-1}}$$

10. Let $f(x) = \ln\left(x^{\sin(x)}\right)$. Find $f'(x)$

$$f(x) = \sin(x)\cdot\ln(x)$$

$$f'(x) = \frac{d}{dx}\left(\sin(x)\cdot\ln(x)\right)$$

$$= \sin(x)\cdot\frac{1}{x} + \ln(x)\cdot\cos(x)$$

$$\boxed{f'(x) = \frac{\sin(x)}{x} + \ln(x)\cdot\cos(x)}$$

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Introduce logarithms when computing certain complicated derivatives

11. Let $f(x) = \sqrt{\dfrac{x^2+4x+4}{x^2-1}}$. Find $f'(x)$.

$$y = \sqrt{\frac{x^2+4x+4}{(x^2-1)}} = \frac{\left((x+2)^2\right)^{\frac{1}{2}}}{(x^2-1)^{\frac{1}{2}}} = \frac{x+2}{(x^2-1)^{\frac{1}{2}}}$$

so

$$\ln(y) = \ln\left(\frac{x+2}{(x^2-1)^{\frac{1}{2}}}\right) = \ln(x+2) - \ln\left((x^2-1)^{\frac{1}{2}}\right)$$

$$\ln(y) = \ln(x+2) - \frac{1}{2}\cdot\ln(x^2-1)$$

so

taking derivatives of both sides

$$\frac{1}{y}\cdot y' = \frac{1}{x+2} - \frac{1}{2}\cdot\frac{1}{x^2-1}\cdot 2x = \frac{1}{x+2} - \frac{x}{x^2-1}$$

$$\boxed{y' = y\left(\frac{1}{x+2} - \frac{x}{x^2-1}\right) = \sqrt{\frac{x^2+4x+4}{x^2-1}}\cdot\left(\frac{1}{x+2} - \frac{x}{x^2-1}\right)}$$

12. Let $f(x) = \Big(\ln(x)\Big)^{\sin(x)}$. Find $f'(x)$.

CANNOT use log properties here

$$y = \left(\ln(x)\right)^{\sin(x)}$$

$$\ln(y) = \ln\left(\left(\ln(x)\right)^{\sin(x)}\right) = \sin(x)\cdot\ln\left(\ln(x)\right)$$

$$\frac{1}{y}\cdot y' = \sin(x)\cdot\frac{d}{dx}\left(\ln(\ln(x))\right) + \ln(\ln(x))\cdot\frac{d}{dx}\left(\sin(x)\right)$$

$$\frac{1}{y}\cdot y' = \sin(x)\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x} + \ln(\ln(x))\cdot\cos(x)$$

$$y' = y\left(\frac{\sin(x)}{x\cdot\ln(x)} + \ln(\ln(x))\cdot\cos(x)\right) = \left(\ln(x)\right)^{\sin(x)}\left(\frac{\sin(x)}{x\cdot\ln(x)} + \ln(\ln(x))\cdot\cos(x)\right)$$

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$P(t) = P_0 \cdot e^{kt}$

Concrete Applications of Derivatives

1. Suppose that a population of bacteria is growing in a petri dish. Suppose also that the first time you look at the dish you count 20 bacteria, and that you count 200 bacteria in the dish 3 hours later. Find a formula for the population as a function of the number of hours t since your first measurement.

   How much time is required for the population to double in size?

$P(t) = 20 \cdot e^{kt}$

$\boxed{P(t) = 20 \cdot e^{\left(\frac{\ln(10)}{3}\right)t}}$

$P(3) = 200 = 20 \cdot e^{k \cdot 3}$

$10 = e^{k \cdot 3}$

$\ln(10) = k \cdot 3$

$k = \frac{\ln(10)}{3}$

find t s.t.

$40 = P(t)$

$40 = 20 \cdot e^{\frac{\ln(10)}{3}t}$

$2 = e^{\frac{\ln(10)}{3}t}$

$\ln(2) = \frac{\ln(10)}{3}t$

$\boxed{t = \frac{3 \cdot \ln(2)}{10}}$.

2. Suppose that Ebola is spreading through the city of Waterbury. Four people are ill two days into the outbreak, and eight people are ill four days in. Find a formula for the number ill a function of days since the outbreak began (0 days in).

   How long until 100 people are ill?

$P(\boxed{2}) = 4 = P_0 \cdot e^{k \cdot \boxed{2}}$

$P(\boxed{4}) = 8 = P_0 \cdot e^{k \cdot \boxed{4}}$

$P_0 = \frac{4}{e^{k \cdot 2}}$ so

$8 = \frac{4}{e^{k \cdot 2}} \cdot e^{k \cdot 4}$

$8 = 4 \cdot \frac{e^{4k}}{e^{2k}}$

$8 = 4 \cdot e^{4k - 2k}$

$8 = 4 \cdot e^{2k}$

$2 = e^{2k}$

$\ln(2) = 2k \Rightarrow k = \frac{\ln(2)}{2}$

$P(t) = P_0 \cdot e^{\frac{\ln(2)}{2}t}$

$P(2) = 4 = P_0 \cdot e^{\frac{\ln(2)}{2} \cdot 2}$

$4 = P_0 \cdot e^{\ln(2)}$

$4 = P_0 \cdot 2$

$P = 2$

$\boxed{P(t) = 2e^{\frac{\ln(2)}{2}t}}$

Solve for t

$100 = 2 \cdot e^{\frac{\ln(2)}{2}t}$

$\Rightarrow \ldots$

$t = \frac{2 \cdot \ln(50)}{\ln(2)}$

3. Suppose that you begin with 100 grams of a radioactive substance. Suppose also that the substance has a half life of 3 years. Find a formula for the amount of radioactive substance remaining after t years.

   What is the weight of the radioactive substance that remains after 9 years?

$P(t) = 100 \cdot e^{kt}$

$\boxed{P(t) = 100 \cdot e^{\left(\frac{\ln(\frac{1}{2})}{3}t\right)}}$

$P(3) = 50 = 100e^{k \cdot 3}$

$\frac{1}{2} = e^{k \cdot 3}$

$\ln(\frac{1}{2}) = k \cdot 3$

$k = \frac{\ln(\frac{1}{2})}{3}$

$P(9) = 100 \cdot e^{\frac{\ln(\frac{1}{2})}{3} \cdot 9}$

$= 100 \cdot e^{\ln(\frac{1}{2}) \cdot 3}$

$= 100 \cdot \left(e^{\ln(\frac{1}{2})}\right)^3$

$= 100 \cdot \left(\frac{1}{2}\right)^3$

$= \frac{100}{8}$

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$\boxed{\text{See solutions to "Related Rates" worksheet}}$

4. An airplane flies directly over a radar station, at a constant altitude of 3 mi above the ground. A little while later, the radar station measures that (a) the distance between the plane and radar station equals 5 mi and (b) that the distance between the plane and radar station is increasing at a rate of 500 mi/hr. What is the ground speed of the airplane at the time of the second measurement?

5. An ice cube melts, with its surface area decreasing at a rate of 3 in^2/s. How fast is the side length decreasing when the side length is 1 in?

6. A streetlight is mounted at the top of a 6 meter pole, and a 2 meter tall person is walking toward it at 2 meters per second. How fast is the length of their shadow changing when they are 4 meters from the streetlight? What about when they are 1 meter from the light?

7. A police officer is walking down a city street, when they spot a wanted felon standing 200 ft away at the corner of the next block. The police officer takes off after the felon at 12 ft/s, and the felon immediately cuts around the corner and runs away at 9 ft/s. What is the rate of change of the distance between the officer and the felon after 10 seconds have passed?

8. Suppose there is a 100 cm long water trough which is empty at time $t = 0$. The cross-section of the trough is an inverted triangle ∇ which is 20 cm across the top, and is 10 cm tall. If the tank is being being filled with water at a constant rate of 400 cm^3/s, how fast is the height changing when the tank is half full?

9. Suppose the water trough above leaks (100 cm long, cross section is a ∇, top $= 20$ cm, and height $= 1$ cm). If water is being added to the tank at a rate of 400 cm^3/s, and is leaking out of the tank at 100 cm^3/s, how fast is the height changing when the tank is half full?

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Abstract Applications of Derivatives

$$L(x) = f'(a)(x-a) + f(a)$$

1. Find a linear approximation for the function $f(x) = \sin(x)$ at $a = \frac{\pi}{4}$.

   Use your answer to approximate $\sin\left(\frac{5\pi}{16}\right)$.

$$f\left(\tfrac{\pi}{4}\right) = \sin\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = \frac{d}{dx}(\sin(x)) = \cos(x)$$

$$f'\left(\tfrac{\pi}{4}\right) = \cos\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\Rightarrow L(x) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$

$$f\left(\tfrac{5\pi}{16}\right) \approx L\left(\tfrac{5\pi}{16}\right) = \frac{\sqrt{2}}{2}\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\cdot\frac{\pi}{16} + \frac{\sqrt{2}}{2}$$

$$\boxed{\sin\left(\tfrac{5\pi}{16}\right) = f\left(\tfrac{5\pi}{16}\right) \approx \frac{\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{32}}$$

2. Find a linear approximation for the function $f(x) = \cos(x)$ at $a = \frac{\pi}{4}$

   Use your answer to approximate $\cos\left(\frac{5\pi}{16}\right)$.

$$f\left(\tfrac{\pi}{4}\right) = \cos\left(\tfrac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f'(x) = -\sin(x)$$

$$f'\left(\tfrac{\pi}{4}\right) = -\sin\left(\tfrac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}$$

$$\Rightarrow L(x) = \frac{-\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}$$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$

$$f\left(\tfrac{5\pi}{16}\right) \approx L\left(\tfrac{5\pi}{16}\right) = \frac{-\sqrt{2}}{2}\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}\cdot\frac{\pi}{16} + \frac{\sqrt{2}}{2}$$

$$\boxed{\cos\left(\tfrac{5\pi}{16}\right) = f\left(\tfrac{5\pi}{16}\right) \approx \frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{32}}$$

3. Find a linear approximation for the function $f(x) = \tan(x)$ at $a = \frac{\pi}{4}$

   Use your answer to approximate $\tan\left(\frac{5\pi}{16}\right)$.

$$f\left(\tfrac{\pi}{4}\right) = \tan\left(\tfrac{\pi}{4}\right) = 1$$

$$f'(x) = \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$f'\left(\tfrac{\pi}{4}\right) = \sec^2\left(\tfrac{\pi}{4}\right) = \frac{1}{\left(\cos\left(\frac{\pi}{4}\right)\right)^2} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{1\cdot 4}{\frac{2}{4}\cdot 4} = \frac{4}{2} = 2$$

$$\Rightarrow L(x) = 2\cdot\left(x - \frac{\pi}{4}\right) + 1$$

Because $\frac{5\pi}{16} \approx \frac{4\pi}{16} = \frac{\pi}{4}$, $f\left(\frac{5\pi}{16}\right) \approx L\left(\frac{5\pi}{16}\right) = 2\left(\frac{5\pi}{16} - \frac{4\pi}{16}\right) + 1$

$$= 2\cdot\frac{\pi}{16} + 1 = 1$$

$$\Rightarrow \boxed{\tan\left(\tfrac{5\pi}{16}\right) \approx 1 + \frac{\pi}{8}}$$

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4. Let $f(x) = x^3 - 6x^2 + 9x + 1$

Find the following. If a requested quantity doesn't exist, answer "DNE".

(a) The intervals where $f(x)$ is increasing/decreasing. Identify which is which.

(b) The intervals where $f(x)$ is concave up/down. Identify which is which.

(c) The x value(s) of the local maxima and local minima of f. Identify which is which.

(d) The x value(s) of the inflection points of f.

$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

$f'(x) = 3(x-1)(x-3)$

[redacted]

(a) f is increasing $\Leftrightarrow$ $f'(x)$ is positive

$(-)(-)$ $(+)$ 1 $(+)(-)$ $(-)$ 3 $(+)(+)$ $(+)$

f is increasing on $(-\infty, 1) \cup (3, \infty)$

f is decreasing on $(1, 3)$

$f''(x) = 6x - 12 = 6(x-2)$

(b) f is concave up $\Leftrightarrow$ f'' is positive

$(-)$ 2 $(+)$

f is concave up [redacted] on $(2, \infty)$

f is concave down on $(-\infty, 2)$

(c) local max at 1

local min at 3

local max/min when change increasing/decreasing

(d) inflection point $\Leftrightarrow$ change concavity

inflection point at $x = 2$

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Use L'Hospital's Rule to answer the following limits. Remember to **show all work**.

This is the best way to learn to do these problems correctly!

5. Does the limit $\lim_{x\to 0+} \frac{\overbrace{\ln(x^3)}^{\to -\infty}}{\underbrace{x^3+3}_{\to 3}}$ converge? If so, what does it converge to?

the limit diverges, but goes to $-\infty$. CANNOT apply L'Hopital's.

$= -\infty$

6. Does the limit $\lim_{x\to\infty} \frac{\overbrace{\ln(x^3)}^{\to \infty}}{\underbrace{x^3+3}_{\to \infty}}$ converge? If so, what does it converge to?

$$\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x^3}\cdot 3x^2}{3\cdot x^2} = \lim_{x\to\infty} \frac{1}{x^3} = 0$$

7. Does the limit $\lim_{x\to\infty} \overbrace{2x}^{\to\infty} \underbrace{\sin\left(\overbrace{\frac{1}{2x}}^{\to 0}\right)}_{\to 0}$ converge? If so, what does it converge to?

$$= \lim_{x\to\infty} \frac{\overbrace{\sin\left(\frac{1}{2x}\right)}^{\to 0}}{\underbrace{\frac{1}{2x}}_{\to 0}} \overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{2x}\right)\cdot \cancel{\frac{-1}{2x^2}}}{\cancel{\frac{-1}{2x^2}}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{2x}\right) = \cos(0) = 1$$

8. Does the limit $\lim_{x\to\infty} \underbrace{\left(1+\frac{8}{x}\right)}_{\to 1}^{\overset{x}{\to\infty}}$ converge? If so, what does it converge to?

indeterminate limit.

$y = \left(1+\frac{8}{x}\right)^x$.

$$\boxed{\ln(y) = \ln\left(\left(1+\frac{8}{x}\right)^x\right) = x\cdot \ln\left(1+\frac{8}{x}\right)}$$

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} x\cdot \ln\left(1+\frac{8}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\overbrace{\ln\left(1+\frac{8}{x}\right)}^{\to 0}}{\underbrace{\frac{1}{x}}_{\to 0}}$$

$$\lim_{x\to\infty} \ln(y) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{1+\overset{\to 0}{\frac{8}{x}}}\cdot\left(\cancel{\frac{-1}{x^2}}\right)\cdot 8}{\cancel{\left(\frac{-1}{x^2}\right)}} = \frac{1\cdot 8}{1} = 8$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = \boxed{e^8}$$

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9. Find the antiderivative of $f(x) = \sin(x)$.

$$F(x) = -\cos(x) + C$$

10. Find the antiderivative of $f(x) = \cos(x)$.

$$F(x) = \sin(x) + C$$

11. Find the antiderivative of $f(x) = \sec^2(x)$.

$$F(x) = \tan(x) + C$$

12. Find the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$. $= x^{-\frac{1}{2}}$

$$F(x) = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C$$

13. Find the antiderivative of $f(x) = \frac{1}{x}$.

$$F(x) = \ln|x| + C$$

14. Find the antiderivative of $f(x) = \frac{1}{x^2}$ $= x^{-2}$

$$F(x) = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$$

15. Find the antiderivative of $f(x) = \frac{1}{x^3}$ $= x^{-3}$

$$F(x) = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$$

16. Suppose $f'(x) = x^2 + 2x + 5$ and that $f(0) = 0$. Find $f(x)$.

$$f(x) = \frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} + 5x + C$$

$$\boxed{\frac{x^3}{3} + x^2 + 5x = f(x)}$$

$$f(0) = 0 = \frac{0^3}{3} + 0^2 + 5 \cdot 0 + C$$

$$0 = C$$

17. Suppose $f'(x) = \frac{1}{x} + x + e^x$ and that $f(1) = 0$. Find $f(x)$.

$$f(x) = \ln|x| + \frac{x^2}{2} + e^x + C$$

$$\boxed{f(x) = \ln|x| + \frac{x^2}{2} + e^x - e - \frac{1}{2}}$$

$$f(1) = \ln|1| + \frac{1^2}{2} + e^1 + C = 0$$

$$0 + \frac{1}{2} + e + C = 0$$

$$C = -e - \frac{1}{2}$$